§1.6 Calculating limits Using the Limit Laws.

Key points: ☆① (Linear) $\boxed{\text{Limit Laws}}$ by general combination

☆☆② Limits by canceling zeros: Factoring technique.

③ Squeeze Theorem.

① Limit Laws:

- Sum / Difference: $\lim\limits_{x\to a}[f(x) \pm g(x)] = [\lim\limits_{x\to a} f(x)] \pm [\lim\limits_{x\to a} g(x)]$
- Constant multiple: $\lim\limits_{x\to a}[c\cdot f(x)] = c\cdot[\lim\limits_{x\to a} f(x)]$, $\lim\limits_{x\to a} c = c$.
- Product: $\lim\limits_{x\to a}[f(x)\cdot g(x)] = [\lim\limits_{x\to a} f(x)]\cdot[\lim\limits_{x\to a} g(x)]$
- Quotient: $\lim\limits_{x\to a}\left[\dfrac{f(x)}{g(x)}\right] = \dfrac{\lim\limits_{x\to a} f(x)}{\lim\limits_{x\to a} g(x)}$ if $\lim\limits_{x\to a} g(x) \neq 0$.
- Power/Root: $\lim\limits_{x\to a}[f(x)]^n = [\lim\limits_{x\to a} f(x)]^n$, $\lim\limits_{x\to a}\sqrt[n]{f(x)} = \sqrt[n]{\lim\limits_{x\to a} f(x)}$.
- Constant / Polynomials: $\lim\limits_{x\to a} c = c$, $\lim\limits_{x\to a} x = a$, $\lim\limits_{x\to a} x^2 = a^2$, $\lim\limits_{x\to a} x^3 = a^3$, ...

  $\lim\limits_{x\to a}\sqrt{x} = \sqrt{a}$ $(a>0)$, $\lim\limits_{x\to a}\dfrac{1}{x} = \dfrac{1}{a}$ $(a\neq 0)$, $\lim\limits_{x\to a}\dfrac{1}{x^2} = \dfrac{1}{a^2}$, ...

eg1. Suppose $\lim\limits_{x\to 2} f(x) = -1$, $\lim\limits_{x\to 2} g(x) = 3$. Compute the following limits.

- $\lim\limits_{x\to 2}\left[5 - \dfrac{x^3}{f(x)+g(x)} + 2\cdot[f(x)]^2\cdot\sqrt{g(x)}\right]$

$$= \lim_{x\to 2}[5] - \lim_{x\to 2}\left[\frac{x^3}{f(x)+g(x)}\right] + \lim_{x\to 2}\left[2\cdot(f(x))^2\cdot\sqrt{g(x)}\right]$$

$$= 5 - \frac{\lim\limits_{x\to 2} x^3}{[\lim\limits_{x\to 2} f(x)] + [\lim\limits_{x\to 2} g(x)]} + 2\cdot[\lim_{x\to 2} f(x)]^2\cdot\sqrt{\lim_{x\to 2} g(x)}$$

$$= 5 - \frac{2^3}{-1+3} + 2\cdot(-1)^2\cdot\sqrt{3}$$

$$= 5 - 4 + 2\cdot\sqrt{3} = \boxed{1 + 2\sqrt{3}}$$ ※

Rmk: All the laws could be applied to one-sided limits.

e.g.2 (Direct plug in)

• $\lim_{x\to 0} \frac{x+1}{3x^2-5x+7} = \frac{0+1}{0+7} = \frac{1}{7}$; • $\lim_{h\to 1} \frac{2h-h^2}{h+1} = \frac{2-1}{1+1} = \frac{1}{2}$; • $\lim_{u\to -3} \sqrt{9-u^2} = \sqrt{9-(-3)^2} = \sqrt{0} = 0$

☆☆② In the quotient form $\lim_{x\to a} \frac{f(x)}{g(x)}$. If both $\lim_{x\to a} f(x) = \lim_{x\to a} g(x) = 0$, then we must cancel out the "zero terms" in $f(x)$ and $g(x)$ by the following factoring technique.

e.g.3. Compute $\lim_{x\to 2} \frac{x^2+x-6}{x^2-2x}$. Remark: If we plug in $x=2$, we get $\frac{2^2+2-6}{2^2-2\cdot 2} = \frac{0}{0}$, which is meaningless.

Solution: $\underset{\text{Factorize}}{=} \lim_{x\to 2} \frac{\cancel{(x-2)}\cdot(x+3)}{x\cdot\cancel{(x-2)}} \underset{\text{cancel out}}{=} \lim_{x\to 2} \frac{x+3}{x} \underset{\text{Plug in}}{=} \frac{2+3}{2} = \boxed{\frac{5}{2}}$.

e.g.4. (S16) If $f(x) = \frac{1}{x+3}$, then $\lim_{x\to 1} \frac{f(x)-f(1)}{x-1} = \lim_{x\to 1} \frac{-1}{4(x+3)} \underset{\text{Plug in}}{=} \frac{-1}{4\cdot 4} = \boxed{-\frac{1}{16}}$

Solution: $\frac{f(x)-f(1)}{x-1} = \frac{\frac{1}{x+3}-\frac{1}{4}}{x-1} = \frac{\frac{4-x-3}{4(x+3)}}{x-1} = \frac{1-x}{4(x+3)(x-1)} = \frac{-1}{4\cdot(x+3)}$ Hint: $\frac{\frac{a}{b}}{c} = \frac{a}{b\cdot c}$

e.g.5. (sec 1.5) Compute $\lim_{x\to -3^+} \frac{x-2}{x^2\cdot(x+3)} = -\infty$. Hint: $x-2 = (-3)-2 = -5$.

$x^2 = (-3)^2 = +9$.

$x+3 = (-3)+3 = 0$

$\left(\frac{-5}{9\cdot(\text{small positive})} = -\infty \right)$ while $x+3$ small but positive since $x\to -3^+$ (approaches -3 from right)

☆ e.g.6. (F16) For what value of c does $\lim_{x\to 2} \frac{cx^2+4}{x-2}$, exist and is finite?

Solution: Notice if we plug in $x=2$, we have $\frac{4c+4}{2-2}$, where the denominator is 0.

So we need the numerator also be zero, i.e., $4c+4=0 \Rightarrow \boxed{c=-1}$

Actually, if $c=-1$, $\lim_{x\to 2} \frac{(-1)\cdot x^2+4}{x-2} = \lim_{x\to 2} \frac{(2+x)\cdot(2-x)}{x-2}$ Hint: $a^2-b^2=(a+b)(a-b)$

$= \lim_{x\to 2} \frac{-(2+x)\cdot\cancel{x}}{\cancel{x}}$ $2^2-x^2 = 4-x^2 = (2+x)(2-x)$

Therefore, $\boxed{c=-1}$ $= -4$.

③ Squeeze Theorem: Suppose $f(x) \le g(x) \le h(x)$ and $\lim_{x\to a} f(x) = \lim_{x\to a} h(x) = L$

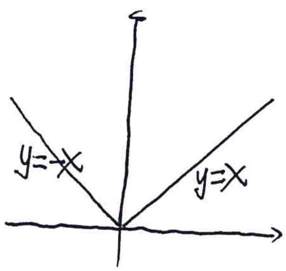
Then $\lim_{x\to a} g(x) = L$

eg7. If $\cos(x) \le g(x) \le 1 - x^2$, find $\lim_{x\to 0} g(x)$.

$$\lim_{x\to 0} \cos x = \lim_{x\to 0} 1 - x^2 = 1 \Rightarrow \lim_{x\to 0} g(x) = 1.$$

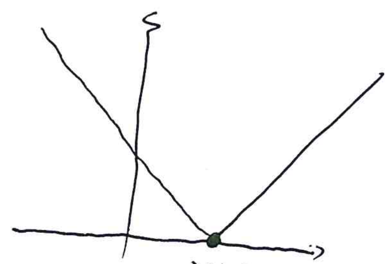
④ Absolute Value Function: $f(x) = |x| = \begin{cases} x & x \ge 0 \\ -x & x < 0 \end{cases}$

graph:



shift: $y = |x - c|$ for some constant c.

$$= \begin{cases} x - c & x \ge c \\ -(x-c) & x < c \end{cases}$$



eg8. Find the limit of $\lim_{x\to 1^+} \frac{2x\cdot(x-1)}{|x-1|}$

$x \to 1^+$ means x approaches 1 from the RIGHT : $x > 1$

$|x-1| = x - 1$

$$\Rightarrow \lim_{x\to 1^+} \frac{2x\cdot(x-1)}{|x-1|} = \lim_{x\to 1^+} \frac{2x\cdot\cancel{(x-1)}}{\cancel{x-1}} = \lim_{x\to 1^+} 2x = 2\cdot 1 = 2$$

§1.8 A, B. Continuity.

Key points: ① Definition and the Graph.

② Continuity of piecewise function

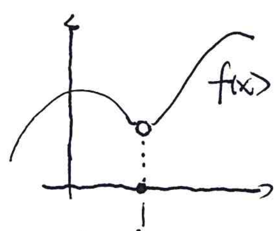
③ Intermediate Value Theorem.

- Definition: $f(x)$ is continuous at $x=a$ if $\lim\limits_{x\to a} f(x) = f(a)$

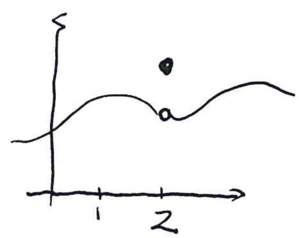
In the graph, it means $y=f(x)$ (the curve) does not have a jump/hole at $x=a$

If it has a jump/hole, $f(x)$ is discontinuous at $x=a$.

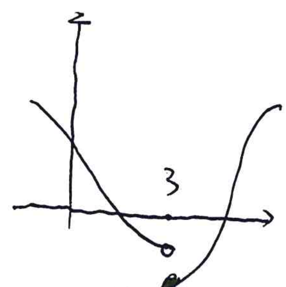
eg1. (Examples of discontinuity). The following functions are discontinuous at $x=a$



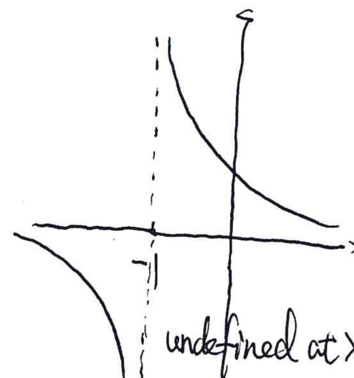
undefined at $x=1$



jump at $x=2$



jump(fall) at $x=3$



undefined at $x=-1$
vertical asymptote

- Domain of continuity. Function defined by formulas are continuous except at those undefined points

eg2. Functions: $y=x$ $\quad y=x^2$ $\quad y=\sqrt{x}$ $\quad y=\frac{1}{x}$ $\quad y=\sin x$

D.O.C.: $(-\infty,+\infty)$ $\quad (-\infty,\infty)$ $\quad [0,\infty)$ $\quad (-\infty,0)\cup(0,\infty)$ $\quad (-\infty,\infty)$

$g(x) = \dfrac{(x^2-3x+1)\cdot\sqrt{x+1}}{x-3}$ $\qquad$ Domain: $\sqrt{x+1} \Rightarrow x+1 \geq 0$, $\quad x-3 \neq 0$

D.O.C.: $[-1,3)\cup(3,+\infty)$

eg3. Let $g(x) = \begin{cases} x^3+2x & \text{if } x \le 5 \\ \dfrac{5x^2-x^3}{x-5} & \text{if } x>5 \end{cases}$. Is $g(x)$ continuous at $x=5$ or not? why?

Remark: $g(x)$ is continuous at $x=5$ if $\lim_{x\to 5} g(x) = g(5)$.

We need to study $\lim_{x\to 5} g(x)$ first.

Solution: $\lim_{x\to 5^-} g(x) = \lim_{x\to 5^-} x^3+2x = 5^3+2\cdot 5 = 135$.

$$\lim_{x\to 5^+} g(x) = \lim_{x\to 5^+} \frac{5x^2-x^3}{x-5} = \lim_{x\to 5^+} \frac{x^2(5-x)}{x-5} \quad \text{cancel the "zeros"}$$

$$= \lim_{x\to 5^+} -x^2 = -25.$$

$\lim_{x\to 5^-} g(x) \neq \lim_{x\to 5^+} g(x)$, therefore, the limit $\lim_{x\to 5} g(x)$ D.N.E.

$g(x)$ is NOT continuous at $x=5$.

eg4. For what value of k will $f(x) = \begin{cases} \dfrac{x^2-3k}{x-3} & \text{if } x\le 2 \\ 8x-k & \text{if } x>2 \end{cases}$ be continuous for all x?

Solution: $\lim_{x\to 2^-} f(x) = \lim_{x\to 2^-} \dfrac{x^2-3k}{x-3} = \dfrac{4-3k}{2-3} = \dfrac{4-3k}{-1} = 3k-4$

$\lim_{x\to 2^+} f(x) = \lim_{x\to 2^+} 8x-k = 16-k$

We need the limit at $x=2$ exists, i.e., $3k-4 = 16-k$.

Solve for k: $4k=20$

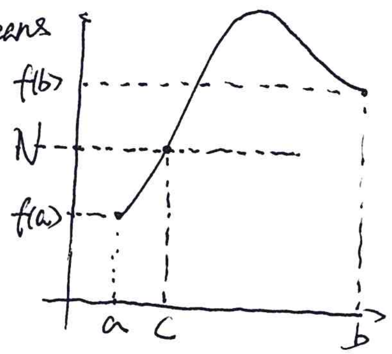
$\Rightarrow \boxed{k=5}$

§1.8 B. ③ Intermediate Value Theorem.

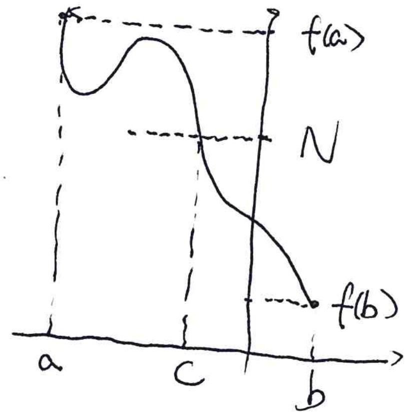
(something in-between)

• (IVT) If f is continuous on $[a,b]$, $f(a) \neq f(b)$, and N is between $f(a)$ and $f(b)$, then there is a $c \in (a,b)$ that satisfies $f(c) = N$.

• Graph: IVT means



$N = f(c)$



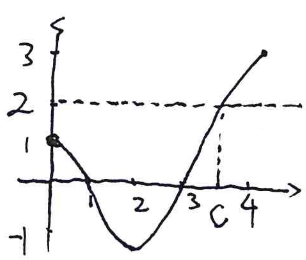
• Rewrite IVT as the following corollary:

In order to solve $f(x) = N$, it is enough to pick a, b such that $f(a), f(b)$, one larger and one smaller than N. Then there is a solution c in (a,b) to $f(x) = N$.

eg1. Let $f(x)$ be continuous with values given below:

x	0	1	2	3	4
$f(x)$	1	0	-1	0	3

Sketch the graph of $y = f(x)$ and find where is c s.t. $f(c) = 2$.



There is a $c \in (3,4)$, such that $f(c) = 2$.

eg2. On which interval must there be a solution to $x^3 - 14 = 36 - 3x$

Consider $f(x) = x^3 - 14 - (36 - 3x)$

$f(3) = 3^3 - 14 - (36 - 3\cdot 3) = 27 - 14 - (36 - 9) = -14 < 0$

$f(4) = 4^3 - 14 - (36 - 3\cdot 4) = 26 > 0$

Therefore, there is a $c \in (3,4)$ such that $f(c) = 0 \Leftrightarrow c^3 - 14 = 36 - 3c$